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SOME PROPERTIES OF POLYNOMIAL SUBGROUP GROWTH GROUPS

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ABSTRACT

A PSG group is one in which the number of subgroups of given index is bounded by a fixed power of this index. The finitely generated PSG groups are known. Here we prove some properties of such groups which need not be finitely generated. We derive, e.g., restrictions on the chief factors (Theorem 1) and on the number of generators of subgroups (Theorem 5).

Let the residually finite group G enjoy the property, that for each natural number n it contains only finitely many subgroups of index n, say $a_n(G)$. Such a group is termed a **polynomial subgroup growth** group (**PSG** group), if the function $a_n(G)$ is polynomially bounded, i.e. there exists an exponent s, such that $a_n(G) \leq n^s$, for each n. The finitely generated PSG groups are determined in [LMS]: they are virtually soluble of finite rank. About non-finitely generated PSG groups it is stated in [LMS]: "*such groups are unlikely to have simple characterisation*". Nevertheless, some properties of these groups are known, and in this paper, which is in the nature of an appendix to [LMS] (and to its predecessors [LM2] and [MS]), we add a few more.

First, applying and extending results of [MS], we give some information about finite chief factors of PSG groups. In the next section we investigate the relationship between the notions of polynomial subgroup growth, finite generation, and

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having bounded rank. This is most conveniently done in the context of profinite groups. We show that a group is PSG **uniformly**, in the sense that all subgroups of finite index satisfy the same inequality for the number of subgroups, if and only if it has bounded upper rank. We end by showing, that even though the number of generators of subgroups of PSG groups need not be bounded, this number increases rather slowly with the index.

NOTATION AND TERMINOLOGY. By a **chief factor** of a group G we mean a group M/N, where M and N are normal subgroups of G, and M/N is a minimal normal subgroup of G/N. If we assume only that M and N are subnormal, but require also N to be a maximal normal subgroup of M, then M/N is a **composition factor**. Finally, if $|G:M|$ and $|G:N|$ are finite, we say that M/N is an **upper** chief (or composition) factor. If M/N is a non-abelian finite chief factor, let $C = C_G(M/N)$. Then G/C is the finite group of automorphisms that G induces on M/N, and it contains a normal subgroup I/C consisting of all the inner automorphisms that M induces on M/N. Then $I/C \cong M/N$, so all non-abelian finite chief factors are isomorphic to upper chief factors. By $d(G)$ we denote the minimal number of generators of G, and the **rank** and **upper rank** of G are

$$\mathrm{rk}(G) = \max(d(H) \mid H \text{ is a finitely generated subgroup of } G),$$
$$\mathrm{urk}(G) = \max(\mathrm{rk}(F) \mid F \text{ is a finite factor group of } G).$$

If G is a profinite group, we mean always by "subgroup" a closed one, and by "generators" a set of topological generators. With these conventions, we define $d(G)$, $\mathrm{rk}(G)$, and PSG as for discrete groups. It is known then that the rank and upper rank of a profinite group are the same, therefore the upper rank of a discrete group is equal to the rank of its profinite completion. We use the term **"prosoluble"** as an abbreviation for **"pro-(finite soluble)"**, i.e. an inverse limit of finite soluble groups. The **core** $\mathrm{Core}_G H$ of a subgroup H of G is the maximal normal subgroup contained in H. An equality containing the sign "=:" serves to define its left hand side.

1. Let G be a group of polynomial subgroup growth. Theorem 4.1 of [MS] states that there exists a number m, depending only on the subgroup growth function of G, such that all upper non-abelian composition factors of G are either sporadic, alternating of degree at most m, or Lie type groups, of Lie rank at most m, and defined over a field of dimension at most m over its prime subfield. Moreover, all upper non-abelian chief factors of G are direct products of at most m simple groups, and G contains a normal subgroup of index at most m, in which all non-abelian upper chief factors are simple.

THEOREM 1: *Let G be a PSG group. Then, in the notation just applied, the number m can be chosen in such a way that the abelian upper chief factors are also direct products of m simple groups at most (i.e. such a chief factor is of order p^k, for some prime p, where $k \leq m$).*

We need some preliminary results.

LEMMA 1.1: *Let G be a finite group whose composition factors are restricted in the same way as in Theorem 4.1 of [MS] (i.e. as detailed before Theorem 1). Then there exists a number t, depending only on m, such that if V is a completely reducible finite faithful module for G, then $|G| \leq |V|^t$.*

This is Corollary 3.3 of [BCP]. Indeed, the assumption that the Lie type composition factors are defined over fields of bounded dimension is unnecessary in this result.

The following is essentially a reformulation of Lemma 1.1.

LEMMA 1.2: *Let G be as in Lemma 1.1. Let V be a finite faithful module for G of characteristic p, and assume that $O_p(G) = 1$. Then $|G| \leq |V|^t$, for the same t as in Lemma 1.1.*

Proof: Let W be the direct sum of the composition factors of V. Then W is a completely reducible G-module, and $|W| \leq |V|$. The kernel of the action of G on W is the stability group (in G) of any composition series of V, and is thus a normal p-subgroup of G, so is trivial, and W is a faithful G-module. Now Lemma 1.1 applies. ∎

Remark: Conversely, for G and V as in Lemma 1.1, we must have $O_p(G) = 1$. Therefore the two Lemmas are equivalent.

Now recall that a group X is **quasi-simple**, if $X' = X$ and $X/Z(X)$ is simple, and that a **semi-simple** group is a central product of quasi-simple ones.

LEMMA 1.3: *Any finite group G has a normal subgroup $F^*(G)$ (the generalised Fitting subgroup) such that*

1. *$C_G(F^*(G)) \leq F^*(G)$.*

2. $F^*(G) = F(G)E(G)$, *where* $F(G)$ *is the Fitting subgroup of* G, *and* $E(G)$ *is semi-simple, or* $E(G) = 1$.

This is Proposition 1.27 of [Go].

LEMMA 1.4: *The Schur multiplier of a finite simple group is generated by two elements.*

See Table 4.1 on p. 302 of the same book.

LEMMA 1.5: *Let* G *be a finite group, and let* $P = O_p(G)$. *If* $P \geq C_G(P)$, *then* $C =: C_G(P/\Phi(P)) = P$.

Proof: If x is a p'-element in C, then x centralises P, so $x = 1$. It follows that C is a normal p-subgroup of G, hence our claim. ∎

Proof of Theorem 1: In view of Theorem 4.1 of [MS], we may assume that the non-abelian upper chief factors of G are simple. Let H/K be an abelian upper chief factor of G, and let L be a normal subgroup of G, which is maximal with respect to the property $H \cap L = K$. Then $T =: G/L$ is a finite group in which $N =: HL/L$ is the unique minimal normal subgroup. Let $|N| = p^n$, for some prime p. We assume that $n > 2$, and we are going to show that T has the following properties:

(i) $O_p(T) \geq C_T(O_p(T))$.

(ii) The rank of $O_p(T)$ is bounded, in terms of the subgroup growth function of G.

(iii) $|T/O_p(T)|$ is bounded, in terms of the same growth function and p.

In particular, property (ii) establishes the Theorem.

Let $F^*(T) = F(T)E(T)$, and write $E = E(T)$, $Z = Z(E)$. By assumption, E/Z is a direct product of simple groups, each of which is normal in T/Z. Let X/Z be one of these, and let $Y = X'$. If Y is simple, then it is a minimal normal subgroup of T, contradicting the uniqueness of N. Therefore $W =: Y \cap Z \neq 1$, and then $N \leq W$, so $n \leq 2$, by Lemma 1.4. This is a contradiction, which shows that $E(T) = 1$. Then $F^*(T) = F(T)$, and the uniqueness of N shows that $F(T) = O_p(T)$. This establishes (i).

Write $P = O_p(T)$, $Q = P/\Phi(P), |Q| = p^k$. By Lemma 1.5, T/P is represented faithfully on Q, so $|T/P| \leq |Q|^t$, for t as in Lemma 1.2. Since Q contains at least $p^{k^2/4}$ subgroups of index $p^{[k/2]}$, the PSG assumption shows that k is bounded. Then the inequality for $|T/P|$ shows (iii).

Now let $r = \mathrm{rk}(P)$. Then we know that P contains a normal subgroup R with r generators such that $|P : R| \leq p^{r(1+\log r)}$ ([LM2], proof of 1.2) and again counting subgroups of R bounds r, thus establishing (ii), and with it our original claim. ∎

2. Any residually finite group G is naturally embedded in a profinite one, its completion, which we denote by $\hat{G}$. These two groups have the same finite quotients. Moreover, $\hat{G}$ is a compact group, and as such has finite Haar measure. We normalise this measure so that $\hat{G}$ has measure 1, and is thus a probability space. But note that the probabilistic argument below can be recast, if one so desires, as a simple counting argument.

THEOREM 2: *Let G be a PSG group. Then there exists a number d, such that all finite factor groups of G can be generated by d elements.*

Proof: The claim is equivalent to $\hat{G}$ being generated (as a topological group) by d elements. Suppose then that $a_n(G) \leq n^s$, for some s. Choose d large enough so that $\Sigma_{n>1} a_n(G)/n^d < 1$. If d elements of $\hat{G}$ do not generate it, there exists some proper subgroup of finite index containing this d-tuple, and the probability of this happening is bounded by the above sum, so the set of d-tuples generating $\hat{G}$ has positive measure, and in particular is not empty, and $\hat{G}$ is generated by d elements. ∎

Note that the choice of d depends only on s.

Remark: More properties of profinite groups which are finitely generated with positive probability, including PSG groups, are given in [Ma].

Definition 1: The group G is of polynomial subgroup growth **uniformly**, if there exist numbers C and s such that $a_n(H) \leq Cn^s$, for all n and all subgroups H of finite index in G.

Definition 2: The profinite group G is of polynomial subgroup growth **uniformly**, if there exist numbers C and s such that $a_n(H) \leq Cn^s$, for all n and all closed subgroups H of G.

For finitely generated residually finite groups, the properties of having a finite rank, having a finite upper rank, and PSG, are equivalent. In general each of these properties implies the next one, but the three properties are distinct, as is clear from the examples in [Se] and [MS]. The situation is somewhat clarified by

THEOREM 3: *Let G be a residually finite group and $\hat{G}$ its profinite completion. The following properties are equivalent:*

1. *G has PSG uniformly.*
2. *G is of finite upper rank.*
3. *$\hat{G}$ is of finite rank.*
4. *$\hat{G}$ has PSG uniformly.*
5. *All subgroups of $\hat{G}$ are PSG.*
6. *$\hat{G}$ is PSG, and either its 2-Sylow subgroup or its 3-Sylow subgroup is finitely generated.*

Proof: The note following Theorem 2 shows that 1. $\Rightarrow$ 2., and it was remarked already that 2. and 3. are equivalent. Let G satisfy 2. and 3., with upper rank r. Then G contains a normal subgroup N of finite index, such that all finite factor groups of N are soluble, and $a_n(G) \leq n^s$, for some s depending only on r and G/N (see Theorem O, Lemma 3.1(i), and Proposition 3.3 of [MS]; also Corollary 4 below). If H has a finite index in G, then $\mathrm{urk}(H) \leq r$, and $H/H \cap N \leq G/N$, so the same bound applies to $a_n(H)$. Therefore 2. implies 1., and a similar reasoning shows that it implies 4., because $\mathrm{rk}(H) \leq \mathrm{rk}(G)$, if H is closed in G.

4. $\Rightarrow$5. is obvious, and 5. $\Rightarrow$6. follows by Theorem 2. Assume 6. Then [DDMS, 6.12] shows that $\hat{G}$ has a normal subgroup N of finite index which has a normal p-complement, for $p = 2$ or 3. If $p = 2$, then N is prosoluble, by the Odd Order Theorem. If $p = 3$, then the only finite simple groups possibly involved in N are Suzuki groups, and by Theorem 4.1 of [MS] only finitely many of these can occur, so by changing N to a smaller subgroup of finite index, if necessary, we get it again to be prosoluble. Then Proposition 3.3 of [MS] shows that 2. holds. ∎

Problem: What is the structure of groups all of whose subgroups are PSG?

Concerning this problem, we may remark that all finitely generated groups which are linear in characteristic 0, and in particular free groups, are contained in profinite PSG groups. This follows from [Lu2], which shows that such a group is a subgroup of a compact p-adic analytic group, and the latter is PSG.

We digress a little to give another application of [DDMS, 6.12].

COROLLARY 4: *Let G be a profinite group, in which either the Sylow 2-subgroup, or both the Sylow 3-subgroup and the Sylow 5-subgroup, are finitely generated.*

Then G is virtually prosoluble.

For $p = 2$ this was already noted, and the other case follows from the fact that the order of the Suzuki groups is divisible by 5, hence a finite group of order prime to both 3 and 5 is soluble.

Proposition 6.12 of [DDMS] and Corollary 4 can be compared with Theorem B of [Lu1], according to which a finitely generated linear group G, for which some Sylow subgroup of G, for whatever prime, is finitely generated, is virtually soluble. Indeed it was remarked by the referee that Corollary 4 can serve to give an alternative proof of this result of [Lu1] for the special cases when the assumptions of Corollary 4 hold. Thus under these assumptions G contains a subgroup of finite index H all of whose finite quotients are soluble. But if G is linear of degree n, then it is residually linear of degree n over finite fields. Now a soluble linear group of degree n has derived length bounded by a function of n only, and it follows that H is soluble with the same bound on its derived length. This proof applies also for characteristics 2 and 3, which are excluded in [Lu1] (see [We] for the requisite results about linear groups).

Returning to PSG groups, we now show that, even though the upper rank need not be finite (as Theorems 6.1 and 6.2 of [MS] show), the number of generators of subgroups of finite index increases rather slowly (recall that we always have $d(H) - 1 \leq |G : H|(d(G) - 1)$).

THEOREM 5: *Let G be a profinite PSG group, and let $d_m(G) = \max(d(H) \mid |G : H| \leq m)$. Then*

$$\lim d_m(G)/\log m = 0.$$

Proof: Suppose that $a_n(G) \leq n^s$, let $\epsilon > 0$, and choose integers k and N so that

$$k > s + 1, \quad s/\log N < \epsilon, \quad \sum_{n>N} n^s/n^k < 1.$$

Let $|G : H| = m$. Then $a_n(H) \leq a_{nm}(G) \leq m^s n^s$. Write $r = 1 + k + [s \log_N m]$. The probability that r elements of H lie in a subgroup of index greater than N is at most $\Sigma_{n>N} m^s n^s/n^r$, which is not more than $\Sigma_{n>N} n^s/n^k < 1$. It follows that some r-tuple of elements of H generates a subgroup of index at most N, and therefore

$$d(H) \leq r + \log N \leq \log N + k + 1 + (s/\log N)\log m \leq \log N + k + 1 + \epsilon \log |G : H|.$$

Thus $\lim d_n(G)/\log n \leq \epsilon$, proving the theorem. ∎

We note that a recent result of Shalev [Sh, 2.4] shows that for pro-p groups the property proved in Theorem 5 implies that G is of finite rank. Our examples show that this is not the case for profinite groups in general.

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